k-String Tension

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based on a work with

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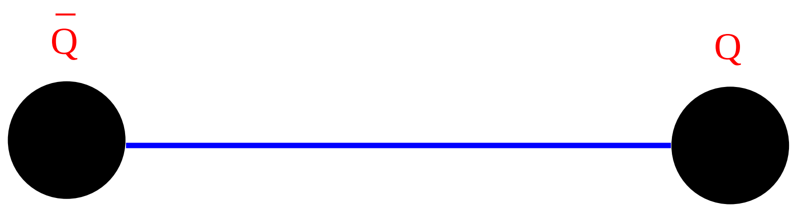
Introduction

In $SU(N)$ gauge theories with adjoint matter (or no matter at all), consider a large Wilson loop in a representation R,

$$W_R = \mathrm{tr} \exp i \int d\tau (A_\mu^a T_R^a \dot{x}^\mu)$$

A large rectangular Wilson loop corresponds to a well-separated quark anti-quark pair in a representation R.



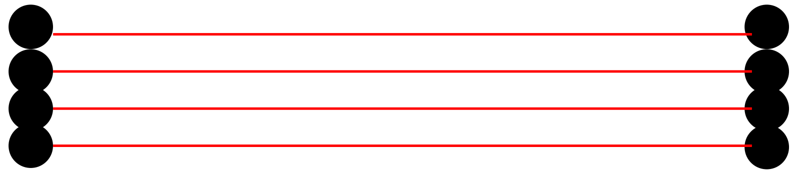
In $d > 2$ we expect

$$\langle W_R \rangle = \exp - \sigma_k \mathcal{A}$$

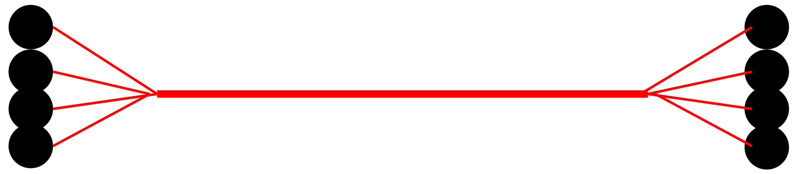
Or equivalently $V(L) = \sigma_k L$. The QCD-string that connects the quark anti-quark pair is called a k-string. The tension σ_k is expected to be a function of the N-ality of the quark. The N-ality is the charge of the quark field under the center of the gauge group $\exp i2\pi k/N$.

An intuitive way of thinking about the k-string is as a bound state of k elementary strings that connect a quark anti-quark pair in the fundamental representation.

If we bring k-elementary strings close to each other



they will attract each other and form a bound state



So, obviously we assume that $\sigma_k < k\sigma_1$. The difference $\sigma_k - k\sigma_1$ is the binding energy.

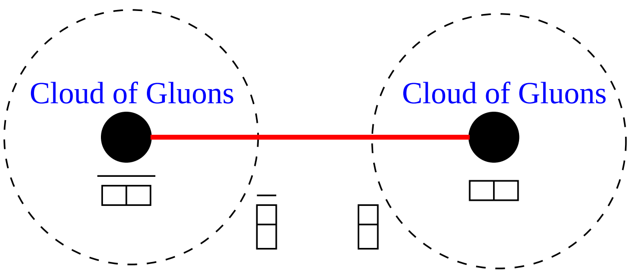
Outline

The outline of this short review is

- Basic properties of the k-string
- Review of previous field theory results
- MQCD and AdS/CFT results
- Eguchi-Kawai reduction
- 3d string tension from 2d
- The massive Schwinger model
- Derivation of the main result
- Summary

Properties of the k-string

The basic property of the k-string is that its tension does not depend on the quark representation, but only on its N-ality, k. This is due to screening: A cloud of gluons should effectively transform the source from its representation to the k-antisymmetric representation.



Two comments are in order:

(i). Screening does not occur in pure YM in $d = 2$.

(ii). In $d > 2$ and large-N screening requires a long time. For example: the "adjoint string" does not break at infinite-N.
$\langle W_{\text{adjoint}} \rangle = \exp -2\sigma_f TL + \frac{1}{N^2} \exp -T/L$, hence $\Lambda T \sim \log N$. In other cases $\Lambda T \sim N^\alpha$.

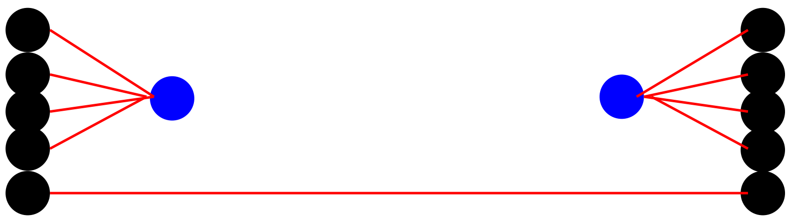
Properties of the k-string

A property of the k-string is the invariance $k \to N - k$. This is due to charge conjugation symmetry. k quarks in the antisymmetric representation are equivalent to $N - k$ antiquarks in the antisymmetric (bar) representation. In particular

$$\sigma_k = \sigma_{N-k}$$

Another property is $k \sim N + k$. Physically, if we place $N + k$ fundamental quarks close to each other, a baryon vertex will break N of them close to the source. Hence, in particular, $\sigma_{N+k} = \sigma_k$.

SU(4) with k=5



Properties of the k-string

Another property is that in the large-N limit $\sigma_k \to k\sigma_1$. The reason is that at infinite N there is no interaction between the fundamental strings. Hence if we bring the strings close to each other they will not form a bound state. Instead we will have a set of k non-interacting fundamental strings.

2d pure YM theory

It is easy to calculate the string tension in 2d pure YM theory, since the pure Yang-Mills theory is free. In the $A_1 = 0$ gauge the theory takes the form

$$S = \int d^2x \left(-\frac{1}{2g^2} \mathrm{tr}(\partial_1 A_0)^2 \right) ,$$

hence the potential is $V = Lg^2 T^a T^a$, namely the string tension is proportional to the quadratic Casimir

$$\sigma_R^{2d} = g^2 C_2(R).$$

Note that since the 2d theory is free there is no screening. For this reason the string tension is *representation dependent*. For example the adjoint string is stable and its tension is non-zero.

The 2d model inspired people to conjecture that in 3d and 4d the main effect of screening is to “reduce” the representation R to the k-antisymmetric and that the string tension in $d > 2$ is proportional to the quadratic Casimir

$$\sigma_k \sim C_2(\text{antisymmetric}) \sim k(N-k).$$

Or more precisely

$$\frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1}.$$

This is called the “Casimir scaling hypothesis”.

Softly broken $\mathcal{N}=2$ super Yang-Mills

Douglas and Shenker considered the k-string in softly broken $\mathcal{N}=2$ super Yang-Mills theory. They used the Seiberg-Witten solution to calculate the k-string tension.

The result is

$$\sigma_k \sim m\Lambda N \sin(\pi \frac{k}{N})$$

The above formula is a first principle result in a 4d confining field theory!

Note that softly broken $\mathcal{N}=2$ SYM is confining, but the confining mechanism is different from what we expect in pure YM theory: $SU(N)$ is broken dynamically to $U(1)^{N-1}$, resulting in "Abelian confinement".

Note also the the QCD k-string in softly broken $\mathcal{N}=2$ SYM is a BPS object.

MQCD

Hanany, Strassler and Zaffaroni considered the k-string in MQCD. They considered a type IIA brane configuration that realizes $\mathcal{N}=1$ super QCD and lifted it to M-theory to obtain a realization of the strongly coupled field theory. The resulting theory is "MQCD". It is a theory that preserves four supercharges and believed to be in the same "universality class" as SQCD. Namely, the interpolation between MQCD and SQCD is made by varying a parameter (x^{10}) and it is assumed that no phase transition occurs in this interpolation.

In MQCD the QCD-string is represented by a membrane that stretches in four-dimensions. It was shown that the membrane energy is $E=\sigma_k L$, such that

$$\frac{\sigma_k}{\sigma_{k'}}=\frac{\sin(\pi\frac{k}{N})}{\sin(\pi\frac{k'}{N})}\,.$$

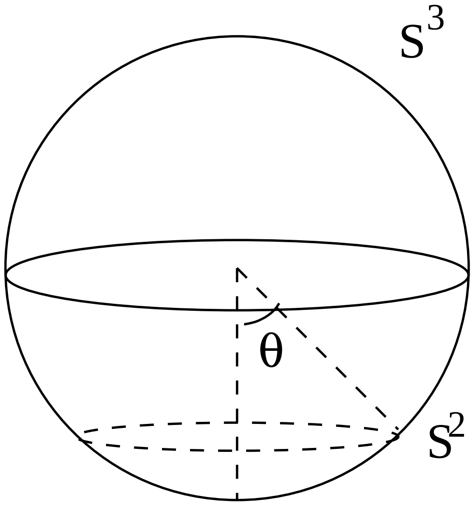
AdS/CFT

Klebanov and Herzog considered the k-string in the AdS/CFT framework. Since the supergravity approximation is valid at infinite N, their results hold when k/N is kept fixed.

They considered two similar (but different) backgrounds: Klebanov-Strassler and Maldacena-Nuñez. Both backgrounds describe in the IR the large-N limit of pure $\mathcal{N} = 1$ SYM (plus extra fields ...).

According to Klebanov and Herzog, when $k \sim N$ the k-string becomes a D3-brane. It is a four dimensional object. The D3 brane wraps an S^2 inside the $R^4 \times S^3$ geometry. So the four dimensional observer sees a two dimensional object. The tension of this object is determined by the angle θ of the S^2 inside the S^3.

AdS/CFT



Their result, exact for MN and approximate (93%) for KS, is that $\theta_k = \pi \frac{k}{N}$ and

$$\sigma_k = \sin\theta_k = \sin\pi\frac{k}{N}\,.$$

Eguchi-Kawai reduction

In 1982 Eguchi and Kawai argued that large-N $SU(N)$ gauge theories admit volume reduction:

If the theory 4d (or 3d) theory is compactified on a tori, certain observables of the theory do not depend on the radii of the tori.

Their paper created excitement in the lattice community, since their result means that one can calculated large-N QCD observables on a single-site lattice. Similarly, it means that the large-N 4d theory could be reduced to a matrix model.

Shortly after the publication of the EK seminal paper, it was shown that a necessary and sufficient condition for the reduction to hold, is that the center of the gauge group, Z_N, is not spontaneously broken.

Unfortunately, pure YM theory undergoes a phase transition called the deconfinement-confinement transition, and the center *is* broken below a critical radius

Eguchi-Kawai reduction

So, it seems that the EK idea is not so useful.

However, few years ago Kovtun, Unsal and Yaffe argued that when the YM theory is coupled to adjoint fermions that admit periodic boundary conditions, there is no phase transition, the theory confines at any radius, and the EK volume reduction could work.

In this talk I would like to use the EK reduction in order to compute the tension of the k-string in a 3d YM theory.

Together with Daniele Dorigoni and Gabriele Veneziano, we considered a 3d YM theory coupled to adjoint matter on $R^2 \times S^1$. We reduced to theory to 2d and calculated the string tension.

3d string tension from 2d

Consider an $SU(N)$ 3d gauge theory coupled to adjoint fermions. Let us compactify the theory on a circle, namely we consider the theory on $R^2 \times S^1$. The Lagrangian of the theory is

$$\sum_{q\in\mathbb{Z}} \mathrm{Tr}\left(-\frac{1}{2g^2}F^{\mu\nu}_{-q}F_{q\,\mu\nu} + (D^\mu\phi)_{-q}(D_\mu\phi)_q \right.$$
$$-\frac{2}{gR}A^\mu_{-q}(D_\mu\phi)_q + \frac{q^2}{g^2R^2}A^\mu_{-q}A_{q\,\mu}+$$
$$+i\bar{\psi}_{-q}\gamma^\mu(D_\mu\psi)_q - \frac{q}{R}\bar{\psi}_{-q}\gamma^2\psi_q$$
$$\left. -g\sum_{q_1}\bar{\psi}_{-q}\gamma^2[\phi_{q_1}.\psi_{q-q1}]\right) ,$$

The claim of EK is that certain quantities are radius independent and that we can take the limit $R \to 0$. The reduced action is

$$S = \int d^2x\, \mathrm{Tr}\left(-\frac{1}{2g^2}F^2_{\mu\nu} + (D_\mu\phi)^2 + \bar{\Psi}i\not{D}\Psi \right.$$
$$\left. +ig[\phi,\bar{\Psi}]\gamma^3\Psi\right).$$

The calculation is carried out in the true vacuum of the theory where the scalar admits a vev $\langle\phi\rangle_{nm} = v_n \delta_{nm} \equiv \frac{(n-(N+1)/2)}{N} \delta_{nm}$.

Note that the above vev respects $\langle P \rangle = 0$, namely the expectation value of the Polyakov loop is zero. It means that the center of the gauge group is not broken - a necessary and sufficient condition for the Eguchi-Kawai reduction.

The string tension in the massive Schwinger model

As a warm-up exercise, let us consider a 2d $U(1)$ gauge theory coupled to a massive fermion. The tension of the string in this model was calculated in 1975 by Coleman, Jackiw and Susskind.

The action of QED2 is

$$S = \int d^2x \left(-\frac{1}{4e^2} F_{\mu\nu}^2 + \bar{\Psi} i \not{\partial} \Psi - m \bar{\Psi} \Psi + Q A_\mu \bar{\Psi} \gamma^\mu \Psi \right)$$

It is convenient to use the equivalent bosonized action

$$S = \int d^2x \left(\frac{1}{2e^2} (\partial_1 A_0)^2 + \frac{1}{2} (\partial_\mu \psi)^2 + m\mu \cos(2\sqrt{\pi}\psi) + \frac{Q}{\sqrt{\pi}} A_0 \partial_1 \psi \right)$$

Now let us add an electron positron pair of charge Q'

$$S' = \int d^2x \, Q' A_0 \left(\delta(x-L) - \delta(x+L)\right)$$

textcolorblueThe string tension in the massive Schwinger model

The action in the presence of the charge, in the limit $L \to \infty$ is

$$S = \int d^2x \, \left(\frac{1}{2}(\partial_\mu \psi)^2 + m\mu \cos(2\sqrt{\pi}\psi) - \frac{e^2}{2}(\frac{Q}{\sqrt{\pi}}\psi + Q')^2 \right)$$

The calculation of the string tension becomes easy when the last term becomes dominant $e^2 \gg m\mu$. The vacuum solution is

$$\psi = -\sqrt{\pi}\frac{Q'}{Q} + \mathcal{O}(\frac{m\mu}{e^2})$$

The vacuum energy is given by

$$E = H - H_0 = m\mu \left(1 - \cos(2\pi \frac{Q'}{Q})\right) 2L$$

where H_0 is the Hamiltonian of the theory without the external charge. The string tension is

$$\sigma = |m|\mu \left(1 - \cos(2\pi \frac{Q'}{Q})\right)$$

The tension vanishes if Q' is a multiple of Q and also when the quark mass is zero.

The string tension in the reduced theory

The relevant part of the reduced action, including the anomaly is

$$S_{eff} = \int d^2x \left\{ \sum_n \frac{1}{2g^2} F_n^2 \right.$$

$$+m \sum_{mn} i(v_n - v_m) \bar{\Psi}_{nm} \gamma^3 \Psi_{mn} +$$

$$\left. \sum_n F_n \left[\frac{-i}{8\pi} \sum_m \ln \left(\frac{\Psi^*_{L\,nm} \Psi_{R\,mn} \;\; \Psi^*_{R\,mn} \Psi_{L\,nm}}{\Psi^*_{L\,mn} \Psi_{R\,nm} \;\; \Psi^*_{R\,nm} \Psi_{L\,mn}} \right) \right] \right\}$$

Let us add a source of the form

$$S' = \int d^2x \left(\frac{k}{2} A^a - \frac{k}{2} A^b \right) \left(\delta(x-L) - \delta(x+L) \right)$$

$$= - \int d^2x \left(\frac{k}{2} F_a - \frac{k}{2} F_b \right)$$

that corresponds to k units of fundamental charge placed at the end of the interval and points along the $SU(N)$ Cartan subalgebra, namely $\vec{k} = (0, ..., 0, k, 0, ..., 0, -k, 0, ...)$.

The string tension in the reduced theory

In order to write the effective action in terms of bosonic fields let us introduce

$$\Psi^*_{L\,nm}\Psi_{R\,mn} = \mu \exp(i\psi_{mn}) \quad ;$$
$$\Psi^*_{R\,mn}\Psi_{L\,nm} = \mu \exp(-i\psi_{nm}) \,.$$

The bosonized action takes the form

$$S_{eff} = \int d^2x \left\{ \sum_n \frac{1}{2g^2} F_n^2 \right.$$
$$-2m\mu \sum_{mn} (v_n - v_m) \sin\psi_{mn} +$$
$$\sum_n F_n \left[\frac{-i}{8\pi} \sum_m \ln\exp\left(2i(\psi_{mn} - \psi_{nm})\right) \right.$$
$$\left.\left. -\frac{k}{2}\delta_n^a + \frac{k}{2}\delta_n^b \right] \right\} \,.$$

Integrating out F_n the Hamiltonian density becomes:

$$
\begin{aligned}
H &= 2m\mu \sum_{mn} (v_n - v_m) \sin\psi_{mn} \\
&+ g^2 \sum_n \left[\frac{1}{4\pi} \sum_m (\psi_{mn} - \psi_{nm}) - \frac{k}{2}\delta^a_n + \frac{k}{2}\delta^b_n \right]^2
\end{aligned}
$$

We could find a solution for the equations of motion in the limit $m/gN \ll 1$.

$$
\begin{aligned}
&\psi_{am} = \pi + \frac{\pi k}{N}\,, \; M_{ma} < 0\,, \\
&\psi_{am} = \frac{\pi k}{N}\,, \; M_{ma} > 0\,, \\
&\psi_{bm} = \pi - \frac{\pi k}{N}\,, \; M_{mb} > 0\,, \\
&\psi_{bm} = -\frac{\pi k}{N}\,, \; M_{mb} < 0\,, \\
&\psi_{mn} = 0\,, \; m, n \neq a, b\,, \\
&\psi_{mn} = -\psi_{nm}\,,
\end{aligned}
$$

where M_{mn} is the mass coefficient multiplying $\sin\psi_{nm}$.

The string tension in the reduced theory

Substituting the solution in the Hamiltonian we obtain

$$\langle H \rangle = 2m\mu \sum_{mn} (v_n - v_m) \sin \psi_{mn} =$$

$$\frac{8m\mu}{N} \sum_{n=0}^{N} |n - \frac{N}{2}| \sin\left(\pi \frac{k}{N}\right)$$

Therefore the string tension is

$$\sigma_k \sim m\mu N \sin\left(\pi \frac{k}{N}\right)$$

or

$$\sigma_k \sim N\lambda_2 \sin\left(\pi \frac{k}{N}\right)$$

where λ_2 is the 2d 't Hooft coupling.

Validity of the result

In our derivation we assumed that the mass term is negligible with respect to the gauge interaction term

$$m\mu N^2 \ll g^2 N^3$$

By using $m \sim \frac{1}{R}$, $\mu \sim g$ and the relation between the 3d gauge coupling and the 2d gauge coupling $g^2 R = g_3^2$, we can rewrite the above condition as

$$\lambda_3 R N \gg 1$$

We now encounter the following difficulty: the masses of the KK modes and the lightest W-bosons are both

$$M_{KK} \sim M_W \sim \frac{1}{RN}$$

Validity of the result

therefore the condition that the mass term is a small perturbation implies that the W-bosons mass (in units of the 3d 't Hooft coupling) goes to zero, namely

$$\frac{M_W}{\lambda_3} \sim \frac{1}{\lambda_3 R N} \ll 1$$

If the W-bosons become massless, the assumption that the dynamics is controlled by the Cartan sub-algebra degrees of freedom may be invalid. Note that it does not invalidate the derivation of the string tension within the 2d framework: within 2d we can always assume a small mass term. The problem is that the Eguchi-Kawai procedure requires a full non-Abelian dynamics and that an "Abelianization" of the problem may not be trusted.

Summary

In this talk I presented a new calculation of the k-string tension, based on the idea of Eguchi-Kawai volume reduction.

The result for the tension in a 3d YM theory coupled to adjoint fermions is

$$\sigma_k \sim N \sin\left(\pi \frac{k}{N}\right)$$

Although the assumption that the 2d theory is controlled by the Cartan subalgebra d.o.f. cannot be justified, our result is consistent with other approaches (MQCD, Douglas-Shenker, Klebanov-Herzog, Armoni-Shifman).

Although the generalization to 4d is technically more complicated, the same result should holds in 4d as well (work in progress).

Based on all the calculations that were carried out so far, I believe that $\mathcal{N}=1$ SYM in 4d in the limit $N \to \infty$, k/N fixed admits a sine formula for the k-string.