Superfluid Fermi Gas: BCS-BEC Crossover and New Material Science

Yoji Ohashi

Department of Physics, University of Tsukuba, Japan
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Experimental Overview</td>
</tr>
<tr>
<td>2.</td>
<td>Fermion and Boson: Difference and Similarity</td>
</tr>
<tr>
<td>3.</td>
<td>Pair formation and Fermi surface effect</td>
</tr>
<tr>
<td>4.</td>
<td>Tunable interaction associated with a Feshbach resonance</td>
</tr>
<tr>
<td>5.</td>
<td>Ground state of a superfluid Fermi gas and BCS-BEC crossover</td>
</tr>
<tr>
<td>6.</td>
<td>Excitations in a Fermi superfluid at T=0 in the BCS-BEC crossover region</td>
</tr>
<tr>
<td>7.</td>
<td>BCS-BEC Crossover theory based on the Coupled Fermion-Boson Model (T=0)</td>
</tr>
<tr>
<td>8.</td>
<td>BCS-BEC crossover at finite temperatures.</td>
</tr>
<tr>
<td></td>
<td>1. Superfluid phase transition</td>
</tr>
<tr>
<td>9.</td>
<td>BCS-BEC crossover at finite temperatures</td>
</tr>
<tr>
<td></td>
<td>2. Superfluid phase below Tc</td>
</tr>
<tr>
<td>10.</td>
<td>Effective interaction between Cooper pairs in the BEC regime</td>
</tr>
<tr>
<td>11.</td>
<td>Effective of a trap potential</td>
</tr>
<tr>
<td>12.</td>
<td>BCS-BEC crossover in a lattice system</td>
</tr>
</tbody>
</table>
Chapter 1. Experimental Overview
Recent Development of cold atom gases

1995

Trap potential

BEC in Bose atom gases

1995

1995

Optical lattice

2002

Mott Transition in Superfluid Bose gases

2002

2002

Feshbach resonance

2004

Superfluid Fermi gas and BCS-BEC crossover

2004

2004

2004
Fermion superfluidity in $^{40}$K Fermi gas


$|9/2, -7/2 > + |9/2, -9/2 >$

$T_F = 0.35 \mu K$

$N = 10^5$

$T_c / T_F \sim 0.08 - 0.2 >> 10^{-4} - 10^{-2} (metal)$

condensate fraction in a K40 Fermi gas

(the number of condensed Cooper pairs)
FIG. 5 (color online). Temperature and magnetic field ranges over which pair condensation was observed (using the same data as in Fig. 4). The right axis shows the range in $T/T_F$ (measured at 1025 G) which was covered. For high degeneracies, fitting $T/T_F$ was less reliable and we regard the condensate fraction as a superior “thermometer.” Note that for an isentropic crossover from a BEC to a Fermi sea, $T/T_F$ is approximately linearly related to the condensate fraction on the BEC side [24]. For our maximum densities the region where $k_F|a| \geq 1$ extends from about 710 G onward.

Single-particle excitations in a superfluid $^6$Li Fermi gas

Fig. 1. RF spectra for various magnetic fields and different degrees of evaporative cooling. The RF offset ($f_0 \times 1 \mu K = h \times 20.8$ kHz) is given relative to the atomic transition $|2\rangle \rightarrow |3\rangle$. The molecular limit is realized for $B = 720$ G (first column). The resonance regime is studied for $B = 822$ G and $B = 837$ G (second and third columns). The data at 875 G (fourth column) explore the crossover on the BCS side. Top row, signals of unpaired atoms at $T^* \approx 6 T_f$ ($T_f = 15 \mu K$); middle row, signals for a mixture of unpaired and paired atoms at $T^* = 0.5 T_f$ ($T_f = 3.4 \mu K$); bottom row, signals for paired atoms at $T^* < 0.2 T_f$ ($T_f = 1.2 \mu K$). The true temperature $T$ of the atomic Fermi gas is below the temperature $T^*$, which we measured in the BEC limit. The solid lines are introduced to guide the eye.

Fig. 2. Measurements of the effective pairing gap $\Delta \nu$ as a function of the magnetic field $B$ for deep evaporative cooling and two different Fermi temperatures, $T_f = 1.2 \mu K$ (solid symbols) and 3.6 $\mu K$ (open symbols). The solid line shows $\Delta \nu$ for the low-density limit, where it is essentially given by the molecular binding energy (25). Inset: The ratio of the effective pairing gaps measured at the two different Fermi energies.

Vortex phase in a $^6$Li Fermi gas

M. Zwierlein, et al., PRL 2005

(This figure shows molecular density profiles.)

The observation of vortices is a clear evidence of “superfluidity” in superfluid Fermi gases.
Collective (surface) mode in a $^6$Li Fermi gas

**FIG. 2** (color online). Oscillations of the radial compression mode at different magnetic fields in the strongly interacting Fermi gas regime. The solid lines show fits by damped harmonic oscillations.

**FIG. 3** (color online). Measured frequency $\Omega_r$ and damping rate $\Gamma_r$ of the radial compression mode, normalized to the trap frequency (sloshing mode frequency) $\omega_r$. In the upper graph, the dashed line indicates $\Omega_r/\omega_r = 2$, which corresponds to both the BEC limit and the collisionless Fermi gas limit. The vertical dotted line marks the resonance position at 837(5) G. The star indicates the theoretical expectation of $\Omega_r/\omega_r = \sqrt{10}/3$ in the unitarity limit. A striking change in the excitation frequency occurs at $\sim 910$ G (arrow) and is accompanied by anomalously strong damping.
Chapter 2.
Fermion and Boson:
Difference and Similarity
Ground State

Free Fermi gas with two hyperfine states

Free Bose gas

Pauli’s exclusion principle (= statistical repulsion)

Bose-Einstein condensation (= statistical attraction)
Occupation of Fermi atoms obeys the Fermi distribution function.

\[ f(\varepsilon - \mu) = \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \]

chemical potential, which is determined from the equation for the number of Fermi atoms:

\[ n = 2 \sum_p f(\varepsilon_p - \mu) \quad (V = 1) \]

At T=0, we find \( \mu = \varepsilon_F \), and

\[ n = 2 \int_0^{\varepsilon_F} d\varepsilon \rho(\varepsilon) = \frac{p_F^3}{3\pi^2} \]

inter-particle distance \( d \sim n^{-1/3} \sim 1/p_F \).
Finite temperatures 1: Fermion

- At finite temperatures, the Fermi edge at $\varepsilon_F$ is smeared in the region $\sim[\varepsilon_F-T, \varepsilon_F+T]$. Thus, effects of the Fermi statistics (giving the step function at $T=0$), become weak when $T \sim T_F$.

- At finite temperatures, $\mu$ deviates from the Fermi energy. At low temperatures, we obtain
  \[ \mu(T) = \varepsilon_F \left[ 1 - \frac{\pi^2}{12} \left( \frac{T}{T_F} \right)^2 \right] \]
  When $\mu < 0$ (which occurs around $T \sim T_F$), $f(\varepsilon - \mu)$ is reduced to the classical Boltzman factor.
  \[ f(\varepsilon - \mu) \sim e^{-\beta|\mu|} e^{-\beta\varepsilon} \]

Namely, the Fermi energy is the characteristic temperature where effects of the Fermi statistical become important.
Occupation of Bose atoms obeys the Bose distribution function.

\[ n_B(\varepsilon - \mu) = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} \]

The chemical potential is determined from the equation for the Number of Bose atoms:

\[ n = \sum_p n_B(\varepsilon_p - \mu) \quad (V = 1) \]

where \( \mu \) must be lower than the lowest energy, \( \mu \leq \varepsilon_0 \). This equation is not satisfied below \( T_c \), where \( T_c \) is the phase transition temperature of the Bose-Einstein condensation. At \( T_c \), \( \mu = 0 \), and

\[ T_c = \frac{2\pi}{(\varsigma(3/2))^{2/3}} \frac{n^{2/3}}{m} \]

\( \varsigma(3/2) = 2.612 \)
Below $T_c$, the condensate fraction $n_0$ is obtained from

$$n = n_0 + \sum_p n_B(\varepsilon_p) \quad \mu = 0$$

In high temperature region, $\mu$ is negatively large, and we obtain the classical Boltzmann distribution function:

$$n_B(\varepsilon - \mu) \sim e^{-\beta |\mu|} e^{-\beta \varepsilon}$$
Similarity between Fermion and Boson

characteristic temperature

\[ \begin{align*}
\text{Fermion} & : T_F \\
\text{Boson} & : T_{BEC}
\end{align*} \]

thermal de Broglie length

\[ \begin{align*}
\text{quantum mechanical size of a particle} \\
\lambda &= \frac{h}{\sqrt{2mT}}
\end{align*} \]

\[ T = \frac{p^2}{2m}, \quad mv = \frac{h}{\lambda} \rightarrow \lambda = \frac{h}{\sqrt{2mT}} \]

Fermion: \( T = T_F \): \( \lambda \sim \frac{1}{p_F} \sim \frac{1}{n^{1/3}} \sim d_F \)

Boson: \( T = T_c \): \( \lambda \sim \frac{1}{n^{1/3}} \sim d_B \)

\( T_F \) and \( T_c \) have the same physics that, at these temperatures, quantum mechanical size of a particle is comparable to the inter-particle distance, and the ‘classical particle picture’ breaks down.
Chapter 3.
Pair formation and
Fermi surface effect
Bose condensation in Fermi gases

Boson

BEC

Fermi degeneracy

Fermion

BEC of molecules (=bosons)
Pair formation 1: two Fermi particle (\( \uparrow, \downarrow \)) case

\[
-\frac{1}{2m} [\nabla_1^2 + \nabla_2^2] \Psi(r_1, r_2) + U(r_1 - r_2) \Psi(r_1, r_2) = E \Psi(r_1, r_2)
\]

\[
\Psi(r_1, r_2) = \sum_k g(k) e^{i k (r_1 - r_2)}
\]

(center of mass momentum =0)

Spin singlet: \( \uparrow \downarrow - \downarrow \uparrow \rightarrow g(-k) = g(k) \)

Spin triplet: \( \uparrow \downarrow + \downarrow \uparrow, \uparrow \uparrow, \downarrow \downarrow \rightarrow g(-k) = -g(k) \)

\[
\frac{k^2}{m} g(k) + \sum_{k'} V_{kk'} g(k') = Eg(k)
\]

\( V_{kk'} = \int d^3r V(r) e^{ik \cdot r} \)

For a contact-type interaction \( (U(r) = -U \delta(r)) \), we obtain

\[
\frac{k^2}{m} - E g(k) = U \sum_{k'} g(k') \quad = \begin{cases} 
0 \quad \text{triplet} \\
C \quad \text{singlet}
\end{cases}
\]

\( E = \frac{k^2}{m} = 2\varepsilon_k \)
Pair formation 1: two Fermi particle (↑,↓) case

Singlet case: \[ 1 = U \sum_k \frac{1}{2\varepsilon_k - E} = U \int d\varepsilon \rho(\varepsilon) \frac{1}{2\varepsilon - E} \]

A bound state is obtained when this equation has a solution with \( E < 0 \). We note that the RHS has a ultraviolet divergence (\( \rho(\varepsilon) \propto \sqrt{\varepsilon} \)). Introducing a cutoff, we find

\[ \frac{1}{U \rho(\omega_c)} = \left[ 1 - \frac{|E|}{2\omega_c} \tan^{-1} \frac{2\omega_c}{|E|} \right] \]

There is a threshold energy to form a molecule in a two particle case.

\[ U \rho(\omega_c) > 1 \]
Pair formation 1: two Fermi particle (↑,↓) case

To avoid the unknown cutoff, we introduce the two-particle s-wave scattering length $a_s$.

$$\frac{4\pi a_s}{m} = \Gamma(0,0) = \frac{1}{2\varepsilon} + \frac{1}{2\varepsilon} + \frac{1}{2\varepsilon} + \ldots$$

$$= -U + (-U) \sum_{p=0}^{\omega_c} \frac{1}{0 - 2\varepsilon} (-U) + \ldots = -\frac{U}{1 - U \sum_{p=0}^{\omega_c} \frac{1}{2\varepsilon}}$$

Then, the bound-state equation can be written as

$$1 = U \sum_{p=0}^{\omega_c} \left[ \frac{1}{2\varepsilon - E} - \frac{1}{2\varepsilon} \right] + U \sum_{p=0}^{\omega_c} \left[ \frac{1}{2\varepsilon} \right]$$

$$1 = \frac{U}{1 - U \sum_{p=0}^{\omega_c} \frac{1}{2\varepsilon}} \sum_{p=0}^{\omega_c} \left[ \frac{1}{2\varepsilon - E} - \frac{1}{2\varepsilon} \right] = -\frac{4\pi a_s}{m} \sum_{p=0}^{\omega_c} \frac{1}{2\varepsilon - E} - \frac{1}{2\varepsilon}$$

This summation converges!
Pair formation 1: two Fermi particle (↑,↓) case

Molecular formation: \( a_s > 0 \).

Bound state energy
\[
E = -\frac{1}{ma_s^2}
\]

\[
\Psi(R = r_1 - r_2) \propto \sum_k \frac{1}{2\varepsilon - E} e^{ik \cdot R} \Rightarrow \frac{1}{\sqrt{2\pi a_s}} \frac{e^{-R/a_s}}{R}
\]

\[
g(k) = \frac{C}{2\varepsilon_k - E}
\]

molecular size = \( a_s \)

Weak U  \[
a_s < 0
\]

Strong U  \[
a_s > 0
\]
Pair formation 2: Cooper problem

many free Fermions (Fermi surface) + two attractively interacting Fermions

\[ -\frac{1}{2m}[\nabla_1^2 + \nabla_2^2] \Psi(r_1, r_2) + U(r_1 - r_2) \Psi(r_1, r_2) = (E + 2\epsilon_F) \Psi(r_1, r_2) \]

\[ \Psi(r_1, r_2) = \sum_{k > k_F} g(k) e^{ik(r_1 - r_2)} \]

Fermion states inside the Fermi surface are occupied.
We obtain the equation for a bound state in the singlet case:

\[ 1 = U \sum_{k > k_F} \frac{1}{2\varepsilon_k - 2\varepsilon_F - E} \]

We introduce a scattering length for the interaction renormalized down to the Fermi level,

\[
\frac{4\pi a_s(\varepsilon_F)}{m} = -\frac{U}{1 - U \sum_{p > k_F} \frac{1}{2\varepsilon}} \left( \frac{4\pi a_s(\varepsilon_F)}{m} = -\frac{V}{1 - V \sum_{p > 0} \frac{1}{2\varepsilon} + V \sum_{p > 0} \frac{1}{2\varepsilon}} = \frac{4\pi a_s}{m} \sum_{p > 0} \frac{1}{2\varepsilon} \right) \]

we obtain

\[ 1 = -\frac{4\pi a_s(\varepsilon_F)}{m} \int_{\varepsilon_F}^{\infty} d\varepsilon \rho(\varepsilon) \left[ \frac{1}{2\varepsilon - 2\varepsilon_F - E} - \frac{1}{2\varepsilon} \right] \]

This equation always has a bound state solution.
Pair formation 2: Cooper problem

weak-coupling case \( |E| \ll \varepsilon_F \)

\[
E = -8 \varepsilon_F e^{k_F a_s (\varepsilon_F)}
\]

\[
\kappa = \frac{1}{k_F} e^{-\frac{\pi}{2 k_F a_s}}
\]

\[
\Psi(R) = \frac{1}{\sqrt{2\pi\kappa}} e^{-R/\kappa}
\]

strong-coupling case \( |E| \ll \varepsilon_F \)

\[
E = -\frac{1}{m a_s (\varepsilon_F)^2} \sim -\frac{1}{m a_s^2}
\]

\[
\kappa = \alpha_s
\]

Fermi surface
+two Fermions

Fermi surface effect (Cooper instability)
Essence of the BCS-BEC crossover phenomenon

\[ T \gg T_c \]
\[ \lambda \ll d \sim 1/ n^{1/3} \]

\[ T = T_c \]
\[ \lambda \sim d \sim 1/ n^{1/3} \]

boson

thermal de Broglie length

BEC
Essence of the BCS-BEC crossover phenomenon

\[ \kappa = \frac{1}{k_F} e^{-\frac{\pi}{2k_F a_s}} \]

weak U

BCS

\[ |k_F a_s| \sim 1 \]

(crossover regime)

strong U

BEC

\[ \kappa = a_s \]

\[ \lambda^{MOL} \sim d_{MOL} \sim (n/2)^{-1/3} \]
Chapter 4. tunable interaction associated with a Feshbach resonance
pairing interaction mediated by boson

- superconductivity
  - Phonon, AF spin fluctuations
- superfluid $^3$He
  - Ferromagnetic spin fluctuations

- superfluid Fermi gas
  - $^{40}$K, $^6$Li
Atomic hyperfine states in the closed channel are different from those in the open channel.

\[
\left| F, F_z \right> = \left| \frac{9}{2}, \frac{-9}{2} \right> + \left| \frac{9}{2}, \frac{-7}{2} \right>
\]

The Zeeman energy of the open-channel is different from the Zeeman energy of the closed channel. This is because of the different magnitude of the electron Bohr magneton and the nuclear one.

\[
\left| F, F_z \right> = \left| \frac{9}{2}, \frac{-9}{2} \right> + \left| \frac{7}{2}, \frac{-7}{2} \right>
\]
Tunable interaction associated with a Feshbach resonance

Fermi atom

molecule

\( V_{\text{eff}} = -g^2 \frac{1}{2\nu} \)

*tunable* by magnetic field

(Timmermans (2001), Holland (2001))

*tunable* pairing interaction

BCS-BEC crossover

(Ohashi and Griffin, PRL 89, 130402 (2002))
Single-channel model and two-channel model

**BCS model (single-channel)**

\[ H = \sum_{p,\sigma} (\varepsilon_p - \mu) c_{p\sigma}^{\dagger} c_{p\sigma} - U \sum_{p, p', q} c_{p+q \uparrow}^{\dagger} c_{p'-q \downarrow}^{\dagger} c_{p \downarrow} c_{p \uparrow} \]

**Coupled fermion-boson model (two-channel)**

\[ H = \sum_{p,\sigma} \varepsilon_p c_{p\sigma}^{\dagger} c_{p\sigma} - U \sum_{p, p', q} c_{p+q \uparrow}^{\dagger} c_{p'-q \downarrow}^{\dagger} c_{p \downarrow} c_{p \uparrow} \]

\[ + \sum_q (\varepsilon_q^B + 2\nu) b_q^{\dagger} b_q + g \sum_{p, q} \left[ b_q^{\dagger} c_{p+q/2 \downarrow}^{\dagger} c_{p+q/2 \uparrow} + c_{p+q/2 \uparrow}^{\dagger} c_{-p+q/2 \downarrow}^{\dagger} b_q \right] \]

\[ N = N_F + 2N_B \Rightarrow H - \mu N \]
\[ U_{\text{eff}} n = U n + (g \sqrt{n})^2 \frac{1}{2 \nu - 2 \mu} \]

crossover region \[ U_{\text{eff}} n \sim \varepsilon_F \]

**narrow F.R.**
\[ g \sqrt{n} < \varepsilon_F \]

\[ \nu \sim \varepsilon_F \]

Boson

Fermion

\[ \nu \gg \varepsilon_F \]

**broad F.R.**
\[ g \sqrt{n} \gg \varepsilon_F \]

Feshbach molecules appear as **real particles** in the crossover region.

Feshbach molecules only appear in the virtual process to produce \( U_{\text{eff}} \).

\((^{40}\text{K}, ^{6}\text{Li})\)

\[ BCS \neq CFB \]

\[ BCS \approx CFB \]
Chapter 5.
Ground state of a superfluid Fermi gas and BCS-BEC crossover
From the Cooper problem to the BCS state

\[ \Psi(r_1, r_2) = \sum_{k} g_k e^{i k \cdot (r_1 - r_2)} |F\rangle \rightarrow \sum_{k} g_k c_{k \uparrow}^\dagger c_{-k \downarrow}^\dagger |F\rangle \]

\[ |F\rangle = \prod_{p < p_F, \sigma} c_{p\sigma}^\dagger |0\rangle = (\sum_{p_1} \tilde{g}_{p_1} c_{p_1 \uparrow}^\dagger c_{-p_1 \downarrow}^\dagger)(\sum_{p_2} \tilde{g}_{p_2} c_{p_2 \uparrow}^\dagger c_{-p_2 \downarrow}^\dagger) \ldots (\sum_{p_{N/2}} \tilde{g}_{p_{N/2}} c_{p_{N/2} \uparrow}^\dagger c_{-p_{N/2} \downarrow}^\dagger) |0\rangle \]

\[ \tilde{g}_p = \theta(p_F - p) \]

\[ |\Psi\rangle = (\sum_{k} g_k c_{k \uparrow}^\dagger c_{-k \downarrow}^\dagger)(\sum_{p_1} \tilde{g}_{p_1} c_{p_1 \uparrow}^\dagger c_{-p_1 \downarrow}^\dagger)(\sum_{p_2} \tilde{g}_{p_2} c_{p_2 \uparrow}^\dagger c_{-p_2 \downarrow}^\dagger) \ldots (\sum_{p_{N/2}} \tilde{g}_{p_{N/2}} c_{p_{N/2} \uparrow}^\dagger c_{-p_{N/2} \downarrow}^\dagger) |0\rangle \]

many Cooper pairs

\[ |\Psi\rangle = (\sum_{p_1} g_{p_1} c_{p_1 \uparrow}^\dagger c_{-p_1 \downarrow}^\dagger)(\sum_{p_2} g_{p_2} c_{p_2 \uparrow}^\dagger c_{-p_2 \downarrow}^\dagger) \ldots (\sum_{p_{N/2}} g_{p_{N/2}} c_{p_{N/2} \uparrow}^\dagger c_{-p_{N/2} \downarrow}^\dagger) |0\rangle \]
BCS ground state

\[ |BCS\rangle = C \prod_k \left[ 1 + g_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right] |0\rangle \]

\[ |\Psi\rangle \] is obtained from ‘N-particle terms’ in \(|BCS\rangle\). Setting \(g=v/u\) and \(C = \prod_k u_k v_k\), we obtain the well known expression,

\[ |BCS\rangle = \prod_k \left[ u_k + v_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right] |0\rangle \]

\[ \langle BCS|BCS\rangle = 1 \Rightarrow u_k^2 + v_k^2 = 1 \]

\(u\) and \(v\) are determined by minimizing the ground state energy \(E_g = \langle BCS|H_{BCS}|BCS\rangle\). The result is \((\xi_p = \varepsilon_p - \mu)\)

\[ u_p^2 = \frac{1}{2} \left( 1 + \frac{\xi_p}{\sqrt{\xi_p^2 + \Delta^2}} \right), \quad v_p^2 = \frac{1}{2} \left( 1 - \frac{\xi_p}{\sqrt{\xi_p^2 + \Delta^2}} \right) \]
superfluid order parameter

\[ \Delta = U \sum_p u_p v_p = U \sum_p \frac{\Delta}{2E_p} \]

\[ E_p = \sqrt{\xi_p^2 + \Delta^2} \]

Gap equation

\[ 1 = U \sum_p \frac{1}{2E_p} \]

\[ 1 = -\frac{4\pi a_s}{m} \sum_p \left[ \frac{1}{2E_p} - \frac{1}{2\varepsilon_p} \right] \] (cutoff-free)

In the usual (weak-coupling) BCS theory, the chemical potential is taken to be equal to \( \varepsilon_F \). However, from general point of view, it is determined by the equation for the number of fermions.

\[ N = 2 \sum_p v_p^2 = \sum_p \left[ 1 - \frac{\xi_p}{E_p} \right] \]
BCS-BEC crossover theory at T=0 (Leggett theory)

\[ 1 = -\frac{4\pi a_s}{m} \sum_p \left[ \frac{1}{2E_p} - \frac{1}{2\epsilon_p} \right] \]

\[ N = 2 \sum_p v_p^2 = \sum_p \left[ 1 - \frac{\xi_p}{E_p} \right] \]

- **Weak coupling BCS limit:** \((k_F a_s)^{-1} \rightarrow -\infty\)
  \[ \Delta = \frac{8}{e^2} \epsilon_F e^{\frac{\pi}{2k_F a_s}}, \quad \mu = \epsilon_F \]

- **Strong-coupling BEC limit:** \((k_F a_s)^{-1} \rightarrow +\infty\)
  \[ \Delta = \sqrt{\frac{16}{3\pi}} |\mu|^{1/4} \epsilon_F^{3/4}, \quad \mu = -\frac{1}{2ma_s^2} < 0 \]

Binding energy of a Cooper pair molecule obtained from the Cooper problem:

\[ E_{bind} = 8\epsilon_F e^{\frac{\pi}{k_F a_s(\epsilon_F)}} \quad \text{(BCS)} \]

\[ E_{bind} = \frac{1}{ma_s^2} \quad \text{(BEC)} \]
BCS-BEC crossover theory at T=0 (Leggett theory)

\[
1 = -\frac{4\pi a_s}{m} \sum_p \left[ \frac{1}{2E_p} - \frac{1}{2\epsilon_p} \right]
\]

\[
N = 2 \sum_p v_p^2 = \sum_p \left[ 1 - \frac{\xi_p}{E_p} \right]
\]

- **Weak coupling BCS limit:** \((k_Fa_s)^{-1} \to -\infty\)

\[
\Delta = \frac{8}{e^2} \epsilon_F e^{\frac{\pi}{2k_Fa_s}}, \quad \mu = \epsilon_F
\]

- **Strong-coupling BEC limit:** \((k_Fa_s)^{-1} \to +\infty\)

\[
\Delta = \sqrt{\frac{16}{3\pi}} \left| \mu \right|^{1/4} \epsilon_F^{3/4}, \quad \mu = -\frac{1}{2ma_s^2} < 0
\]

Binding energy of a Cooper pair molecule obtained from the Cooper problem:

\[
E_{bind} = 8\epsilon_F e^{k_Fa_s(\epsilon_F)} \quad (BCS) \quad E_{bind} = \frac{1}{ma_s^2} \quad (BEC)
\]
BCS-BEC crossover at $T=0$ (self-consistent solution)

\[ \frac{\Delta}{\varepsilon_F} \]

\[ \frac{\mu}{\varepsilon_F} \]

$\Delta / \varepsilon_F$

$\mu / \varepsilon_F$

BCS $\leftrightarrow (k_F a_s)^{-1} \leftrightarrow$ BEC
Fermi superfluid = molecular BEC?

(1) superfluid order parameter

$$\Delta = U \sum_p \langle BCS | c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} | BCS \rangle$$

(2) ground state energy

**BEC regime:** $E_G = \langle BCS | H | BCS \rangle = - | \mu | N = - \frac{1}{ma_s^2} \frac{N}{2} = -E_{bind} N_B$

All atoms form Cooper pairs with $E_{bind}$.

**BCS regime:** $E_G - E_N = - \frac{1}{2} \rho(\varepsilon_F) \Delta^2 = - \frac{3}{8} \left( N \frac{\Delta}{\varepsilon_F} \right) \Delta$

Only a small fraction ($N \times (\Delta/\varepsilon_F)$) of atoms form Cooper pairs with $E_{bind}=\Delta$.

→ The molecular picture is not good in the BCS regime.
Fermi superfluid = molecular BEC?

\[ |BCS\rangle = \prod_k \left[ u_k + v_k \ c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right] |0\rangle \]

\[ u_k \sim 0, v_k \sim 1 \iff \varepsilon_p < \varepsilon_F - \Delta \]

- Energy levels are fully occupied deep inside the Fermi level (as in the case of a free Fermi gas \(|F\rangle = \prod_{p\sigma} c_{p\sigma}^\dagger |0\rangle\)). These occupying atoms do not contribute to the condensation energy.

- Cooper pairs near the Fermi surface (\(|\xi_p| < \Delta\)) only contribute to the condensation energy.

\[ N_{pair} \sim N \times \frac{\Delta}{\varepsilon_F} \]
Fermi superfluid = molecular BEC?

(3) molecular picture in the BCS regime

\[ n_p = \langle BCS | c_{p\sigma}^+ c_{p\sigma} | BCS \rangle = v_p^2 = \frac{1}{2} \left[ 1 - \frac{\epsilon_p}{\sqrt{(\epsilon_p - \epsilon_F)^2 + \Delta^2}} \right] \]

size of a Cooper pair \( \xi \) \( \delta p \sim 1/\xi \) \( \epsilon_F \sim (p_F \pm \delta p)^2/2m \)

uncertainty principle

\[ \xi \sim \frac{k_F}{m\Delta} = \frac{1}{k_F} e^{-\frac{\pi}{2k_F a_s}} = \kappa \]
Fermi superfluid = molecular BEC?

(3) **condensate fraction**

= the number of Bose-condensed Cooper pair molecules

\[
|BCS\rangle = e^{ \sum g_k c_{p,\uparrow}^+ c_{-p,\downarrow}^+ } |0\rangle
\]

\[
|\Phi\rangle = e^{ \sqrt{n_0 b_0^+} } |0\rangle \quad \text{(BEC ground state)}
\]

\[
H = \sum_q \left( \frac{q^2}{2m} - \mu \right) b_q^+ b_q + \frac{U}{2} \sum_{p,p',q} b_{p+q}^+ b_{p'^{-q}}^+ b_p b_p
\]

\[
b_0 \rightarrow \langle b_0 \rangle \quad \text{(BEC order parameter)}
\]

\[
n_0 = \langle b_0 \rangle^2 \quad \text{(condensate fraction in BEC)}
\]

\[
b_0 = \sum_k g_k c_{k,\uparrow}^+ c_{-k,\downarrow}^+ = \sum_k \frac{\nu_k}{u_k} c_{k,\uparrow}^+ c_{-k,\downarrow}^+
\]
Fermi superfluid = molecular BEC?

\[ [b_0, b_0^\dagger] = \sum_k g_k^2 (1 - c_k^\dagger c_k^\dagger - c_{-k}^\dagger c_{-k}^\dagger). \]

Within the **expectation value** in terms of \(|\text{BCS}>\), this commutation relation is evaluated as

\[ [b_0, b_0^\dagger] = \sum_k g_k^2 (1 - 2v_k^2), \quad B_0 = \frac{1}{\sqrt{\sum_k g_k^2 (1 - 2v_k^2)}} b_0, \quad [B_0, B_0^\dagger] = 1. \]

When RHS is positive, **Boson**!

This condition is always satisfied when \(\mu<0\) (\(v^2<0.5\)). This is realized in the **strong-coupling BEC regime**, where the molecular character is OK. In this case, we have

\[ |\text{BCS}> = e^{\sum g_k c_{p_k}^\dagger c_{-p_k}^\dagger} |0> = e^{\sqrt{n_0}b_0^\dagger} |0> \]

\[ n_0 = \sum_k g_k^2 (1 - 2v_k^2) = \sum_k \left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}}\right)^2 \frac{\xi_k}{E} = \frac{N}{2} \]
Fermi superfluid = molecular BEC?

\[ [b_0, b_0^\dagger] = \sum_k g_k^2 (1 - 2v_k^2) \]

\[ B_0 = \frac{1}{\sqrt{\sum_k g_k^2 (1 - 2v_k^2)}} b_0, \quad [B_0, B_0^\dagger] = 1. \]

In the BCS regime (\(\mu > 0\)), RHS is negative and one cannot introduce such an approximate boson operator.

In this regime, Bose-condensed pairs consist of particle pairs and hole pairs due to the presence of the Fermi surface.
Fermi superfluid = molecular BEC?

$$|BCS\rangle = \prod_k \left[ u_k + v_k c_{k\uparrow}^+ c_{-k\downarrow}^+ \right] |0\rangle = \prod_{\varepsilon<\mu} \left[ u_k c_{-k\downarrow} c_{k\uparrow} + v_k \right] \prod_{\varepsilon>\mu} \left[ u_k + v_k c_{k\uparrow} c_{-k\downarrow}^+ \right] |F(\mu)\rangle$$

$$|F(\mu)\rangle = \prod_{\varepsilon<\mu} c_{k\uparrow}^+ c_{-k\downarrow}^+ |0\rangle \quad \text{('Fermi vacuum')}$$

$$\begin{align*}
\sum u_k c_{-k\downarrow} c_{k\uparrow} + \sum_{\varepsilon>\mu} v_k c_{k\uparrow}^+ c_{-k\downarrow}^+ \\
= e^{\varepsilon<\mu} v_k + \sum_{\varepsilon>\mu} u_k c_{k\uparrow}^+ c_{-k\downarrow}^+
\end{align*}$$

$$|F(\mu)\rangle = e^{\sqrt{n_0} B_0^\dagger} |F(\mu)\rangle$$

$$[b_0, b_0^\dagger] = -\sum_{\varepsilon<\mu} \frac{u_k^2}{v_k^2} (1 - 2v_k^2) + \sum_{\varepsilon>\mu} \frac{v_k^2}{u_k^2} (1 - 2v_k^2) > 0 \quad \text{The first term is taken to be equal to 0 when } \mu<0.$$}

$$B_0^\dagger = \frac{1}{\sqrt{-\sum_{\varepsilon<\mu} \frac{u_k^2}{v_k^2} (1 - 2v_k^2) + \sum_{\varepsilon>\mu} \frac{v_k^2}{u_k^2} (1 - 2v_k^2)}} b_0^\dagger, \quad [B_0, B_0^\dagger] = 1.$$}

$$n_0 = -\sum_{\varepsilon<\mu} \frac{u_k^2}{v_k^2} (1 - 2v_k^2) + \sum_{\varepsilon>\mu} \frac{v_k^2}{u_k^2} (1 - 2v_k^2) = \frac{N}{2} \left( \frac{\Delta}{\varepsilon_F} \right)$$

BCS limit
Fermi superfluid = molecular BEC?

condensate fraction

\[ \frac{n_0}{N} \]

ODLRO

\( (k_F a_s)^{-1} \)

BCS BEC

condensed molecular bosons

Particle pairs

Particle pairs

Hole pairs

(Fermi surface effect)
In the BEC phase, the condensate fraction is given by the maximum O(N) eigen-value of this density matrix:

\[ \rho_1 = \Phi^+(r_1)\Phi^+(r_2) \Rightarrow \Phi^+(r_1)\Phi^+(r_2) \quad | r_1 - r_2 | \rightarrow \infty \]

(1) Boson BEC

\[ \rho_1 = n_0 \phi^*(r_1)\phi(r_2) + O(1) \]

normalized eigen function of \( \rho \)

\[ n_0 = \int dr |\langle \Phi (r) \rangle|^2 \]
Condensate fraction (ODLRO)

(2) Fermi superfluid

$$\rho_2 = \langle \Psi_{\uparrow}(r_1)\Psi_{\downarrow}(r_2)\Psi_{\downarrow}(r_3)\Psi_{\uparrow}(r_4) \rangle \Rightarrow \langle \Psi_{\uparrow}(r_1)\Psi_{\downarrow}(r_2) \rangle \langle \Psi_{\downarrow}(r_3)\Psi_{\uparrow}(r_4) \rangle$$

$$| (r_1, r_2) - (r_3, r_4) | \to \infty$$

$$\rho_2 = n_0 \phi^*(r_1, r_2) \phi(r_3, r_4) + O(1)$$

$$n_0 = \iint dr_1 dr_2 \left| \langle \Psi_{\uparrow}(r_1)\Psi_{\downarrow}(r_2) \rangle \right|^2$$

Uniform Fermi superfluid at T=0 (|BCS >)

$$\Psi_{\sigma}^+(x) = \sum_k e^{-ik \cdot r} c_{k\sigma}^+$$

$$\langle BCS | \Psi_{\uparrow}(r)\Psi_{\downarrow}(r') | BCS \rangle = \sum_k e^{-ik \cdot (r-r')} \langle BCS | c_{k\uparrow}^+ c_{-k\downarrow}^+ | BCS \rangle = \sum_k u_k v_k e^{-ik \cdot (r-r')}$$

$$n_0 = \sum_k u_k^2 v_k = \sum_k \frac{\Delta^2}{4E_k^2}$$

$$\left( n_0 = -\sum_{\varepsilon < \mu} u_k^2 (1 - 2v_k^2) + \sum_{\varepsilon > \mu} v_k^2 (1 - 2u_k^2) \right)$$
Chapter 6.
Excitations in a Fermi superfluid at T=0 in the BCS-BEC crossover region
Excitations in Fermi and Bose superfluids

<table>
<thead>
<tr>
<th></th>
<th>collective modes</th>
<th>single-particle excitations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bose atom BEC</td>
<td>○</td>
<td>×</td>
</tr>
<tr>
<td>Fermi superfluid</td>
<td>○</td>
<td>○</td>
</tr>
</tbody>
</table>

Cooper-pair is a molecular Boson with a finite binding Energy $\Delta$. 
1. Single particle excitations at $T=0$

Leggett’s theory

Since thermal fluctuations are absent at $T=0$, we can use the mean-field theory in the BCS-BEC crossover region, including the equation of states to take into account the strong-coupling effects on $\mu$.

$$H = \sum_{p\sigma} (\varepsilon_p - \mu) c_{p\sigma}^{\dagger} c_{p\sigma} - \Delta \sum_{p} (c_{p\uparrow}^{\dagger} c_{-p\downarrow}^{\dagger} + c_{-p\downarrow} c_{p\uparrow})$$

**Bogoliubov transformation**

$$
\begin{pmatrix}
    c_{p\uparrow}^{\dagger} \\
    c_{-p\downarrow}^{\dagger}
\end{pmatrix}
= 
\begin{pmatrix}
    u_p & v_p \\
    -v_p & u_p
\end{pmatrix}
\begin{pmatrix}
    \alpha_{p\uparrow} \\
    \alpha_{-p\downarrow}
\end{pmatrix}
$$

Bogolunon (Fermion)

$$
H = \sum_{p\sigma} E_p \alpha_{p\sigma}^{\dagger} \alpha_{p\sigma}
$$

$$
E_p = \sqrt{(\varepsilon_p - \mu)^2 + \Delta^2}
$$

Bogoliubov single-particle excitation spectrum

$$
|0\rangle = |BCS\rangle \quad (\forall \alpha_{p\sigma} |0\rangle = 0)
$$
1. Single particle excitations at $T=0$

Energy gap $E_g = \left\{ \begin{array}{ll} \Delta \\
\sqrt{\mu^2 + \Delta^2} \\
|\mu| 
\end{array} \right.$

BCS: $\mu > 0$

BEC: $\mu < 0$

BEC limit ($|\mu| >> \Delta$)

Note: The **binding energy** of a Cooper-pair is given by $2E_g$. 

$$E_p = \sqrt{(\varepsilon_p - \mu)^2 + \Delta^2}$$
Single-particle excitations can be seen in the superfluid density of states.

\[ N(\omega) = \sum \delta(\omega - E_p) = \int_0^\infty d\varepsilon \rho(\varepsilon) \delta(\omega - E) \]

The coherence peak is absent in the excitation spectrum in the BEC regime.

\[ N(\omega) = \rho(\mu) \frac{\omega}{\sqrt{\omega^2 - \Delta^2}} \theta(\omega - \Delta) \]
1. Single particle excitations at $T=0$ (BEC regime)

Excitation spectrum in the BEC regime is simply given by the density of states of a free Fermi gas.

$$N(\omega) = \rho(\omega+ | \mu |)$$
1. Single particle excitations at T=0 (rf spectroscopy)

Single-particle excitations can be observed by using the rf-tunneling current spectroscopy.

Fig. 1. RF spectra for various magnetic fields and different degrees of evaporative cooling. The RF offset ($\delta_B = 20.8$ kHz) is given relative to the atomic transition $|2\rangle \rightarrow |3\rangle$. The molecular limit is realized for $B = 720$ G (first column). The resonance regime is studied for $B = 822$ G and $B = 837$ G (second and third columns). The data at 875 G (fourth column) explore the crossover on the BCS side. Top row, signals of unpaired atoms at $T = T_F$ ($T_F = 15$ μK); middle row, signals for a mixture of unpaired and paired atoms at $T = 0.27 T_F$ ($T_F = 3.4$ μK); bottom row, signals for paired atoms at $T = T_F$ ($T_F = 1.2$ μK). The true temperature $T$ of the atomic Fermi gas is below the temperature $T_F$, which we measured in the BEC limit. The solid lines are introduced to guide the eye.
2. collective excitations at T=0 (1. Boson BEC)

**Bogoliubov mode in a boson BEC**

\[
H = \sum_{q} \left( \frac{q^2}{2m} - \mu \right) b_q^{+} b_q + U \sum_{p,p',q} b_{p+q}^{+} b_{p'-q}^{+} b_p b_p
\]

\[
b_0 \to \langle b_0 \rangle = \sqrt{n_0}, \quad + \text{Bogoliubov approximation}
\]

\[
H = E_g + \sum_{q} \left( \frac{q^2}{2m} - \mu + 2Un_0 \right) b_q^{+} b_q + \frac{Un_0}{2} \sum_{p} \left[ b_p^{+} b_{-p}^{+} + b_{-p} b_p \right]
\]

\[
\mu \text{ is chosen so that the ground state energy } E_g = \frac{U}{2} n_0^2 - \mu n_0 \text{ can be minimum, which gives } \mu = Un_0.
\]

\[
H = E_g + \sum_{q} \left( \frac{q^2}{2m} + Un_0 \right) b_q^{+} b_q + \frac{Un_0}{2} \sum_{p} \left[ b_p^{+} b_{-p}^{+} + b_{-p} b_p \right]
\]

**Bogoliubov transformation**
2. collective excitations at T=0 (1. Boson BEC)

Bogoliubov mode in a boson BEC

\[ H = \sum_q E_q \beta_q^+ \beta_q \]

\[ E_q = \sqrt{\varepsilon_q (\varepsilon_q + 2Un_0)} \]

\[ = \sqrt{\frac{U}{m} n_0} q = v_\phi q \quad (q \to 0) \quad \text{gapless Bogoliubov phonon} \]

This Goldstone mode is associated with the spontaneous breakdown of the continuous gauge symmetry in the BEC phase.

\[ b_0 \to \langle b_0 \rangle = \sqrt{n_0} e^{i\theta} \]

The BEC is dominated by collective excitations only.
2. collective excitations at $T=0$ (2. Superfluid Fermi gas)

\[ |BCS\rangle = \prod (u_k + v_k c^+_k c^-_{-k}) |0\rangle \]

\[ \Delta \rightarrow \Delta e^{i\theta} \]

\[ b_0 \rightarrow \langle b_0 \rangle = \sqrt{n_o} e^{i\theta} \]

\[ |BCS'\rangle = \prod (u_k + v_k c^+_k c^-_{-k} e^{i\theta}) |0\rangle. \]

When the order parameter oscillates with $q$, the additional pair amplitude, $\langle \phi_q \rangle = \langle c_{-p+q/2} c_{p+q/2} \rangle$ and $\langle \phi^+_q \rangle = \langle c^+_{p+q/2} c^+_{-p+q/2} \rangle$ are induced. The resulting mean-field Hamiltonian is

\[ H = H_{BCS} - U \sum_{p,q} (\langle \phi^+_{q} \rangle c_{-p+q/2} c_{p+q/2} + \langle \phi_q \rangle c^+_{p+q/2} c^+_{-p+q/2}) \]

This perturbation again generates the oscillation of the order parameter. In the linear response theory, we obtain
2. collective excitations at $T=0$ (2. Superfluid Fermi gas)

\[
\begin{align*}
\langle \phi_q(t) \rangle &= -U \int_0^\infty dt' \langle \langle \phi_q(t); \phi_q^\dagger(t') \rangle \rangle \langle \phi_q(t') \rangle - U \int_0^\infty dt' \langle \langle \phi_q(t); \phi_q(t') \rangle \rangle \langle \phi_q^\dagger(t') \rangle \\
\langle \phi_q^\dagger(t) \rangle &= -U \int_0^\infty dt' \langle \langle \phi_q^\dagger(t); \phi_q^\dagger(t') \rangle \rangle \langle \phi_q(t') \rangle - U \int_0^\infty dt' \langle \langle \phi_q^\dagger(t); \phi_q(t') \rangle \rangle \langle \phi_q^\dagger(t') \rangle
\end{align*}
\]

Here,

\[
\langle \langle \phi_q(t); \phi_q^\dagger(t') \rangle \rangle = -i \theta(t-t') \langle \text{BCS} \vert [\phi_q(t'), \phi_q^\dagger(t')] \vert \text{BCS} \rangle
\]

In the frequency space $\Omega$,

\[
\begin{align*}
\langle \phi_q(\Omega) \rangle &= -U \langle \langle \phi_q; \phi_q^\dagger \rangle \rangle \langle \phi_q(\Omega) \rangle - U \langle \langle \phi_q; \phi_q \rangle \rangle \langle \phi_q^\dagger(\Omega) \rangle \\
\langle \phi_q^\dagger(\Omega) \rangle &= -U \langle \langle \phi_q^\dagger; \phi_q^\dagger \rangle \rangle \langle \phi_q(\Omega) \rangle - U \langle \langle \phi_q^\dagger; \phi_q \rangle \rangle \langle \phi_q^\dagger(\Omega) \rangle
\end{align*}
\]
2. collective excitations at T=0 (2. Superfluid Fermi gas)

Mode equation:

\[
0 = \begin{vmatrix}
1 + U \left\langle \phi_q \phi_q^\dagger \right\rangle_\Omega & -U \left\langle \phi_q \phi_q \right\rangle_\Omega \\
-U \left\langle \phi^{\dagger}_q \phi_q \right\rangle_\Omega & 1 + U \left\langle \phi^{\dagger}_q \phi_q \right\rangle_\Omega
\end{vmatrix}
\]

\[
1 + \frac{U}{2} \Pi_{22}(q, \Omega) = \left( \frac{U}{2} \right)^2 \Pi_{12}(q, \Omega) \frac{1}{1 + \frac{U}{2} \Pi_{11}(q, \Omega)}
\]

\[
\begin{align*}
\Pi_{22}(q, \Omega) &= -\sum_p \left[ 1 + \frac{\xi_+ \xi_- + \Delta^2}{E_+ E_-} \right] \frac{E_+ + E_-}{(E_+ + E_-)^2 - \Omega^2} \\
\Pi_{11}(q, \Omega) &= -\sum_p \left[ 1 + \frac{\xi_+ \xi_- - \Delta^2}{E_+ E_-} \right] \frac{E_+ + E_-}{(E_+ + E_-)^2 - \Omega^2} \\
\Pi_{12}(q, \Omega) &= -\Pi_{21}(q, \Omega) = \sum_p \left( \frac{\xi_+}{E_+} + \frac{\xi_-}{E_-} \right) \frac{i\Omega}{(E_+ + E_-)^2 - \Omega^2}
\end{align*}
\]

- phason
- ampliton
- coupling mode
2. collective excitations at $T=0$ (2. Superfluid Fermi gas)

In the long wave length limit, taking $\Omega = v_\phi q$, we obtain the sound velocity in the entire BCS-BEC crossover at $T=0$,

$$v_\phi = \sqrt{\frac{1}{m}} \sqrt{\sum \frac{\Delta^2}{E_p^5} \varepsilon_p + \sum \frac{\xi_p}{2E_p^3}} \sum \frac{1}{E_p^3} + \left( \sum \frac{\xi_p}{E_p^3} \right)^2 / \sum \frac{\Delta^2}{E_p^3}$$

- **BCS regime:** $v_\phi = \frac{1}{\sqrt{3}} v_F$  
  *Anderson-Bogoliubov mode*

- **BEC regime:** $v_\phi = \sqrt{\frac{U_B n_0}{M}}$  
  *Bogoliubov phonon*

$$U_B = \frac{4\pi a_B}{M} = \frac{4\pi (2a_s)}{M} > 0 \quad (M = 2m)$$

Effective *repulsive* interaction, given by $a_B = 2a_s$, looks working between Cooper-pair molecules in the BEC regime.
2. collective excitations at $T=0$ (2. Superfluid Fermi gas)

\[ v_\phi = \frac{1}{\sqrt{3}} v_F \]
\[ v_\phi = \sqrt{\frac{U_B n_0}{M}} \]

**NOTE:**
1. In a charged Fermi superfluid, this sound mode remains only just below $T_c$ (Carlson-Goldman mode). At $T=0$, the plasma only exists.
2. In a more sophisticated theory, it has been pointed out that the effective interaction between molecules is given by $a_B = 0.6 a_s$. 
Chapter 7.
BCS-BEC Crossover theory based on the Coupled Fermion-Boson Model (T=0)
BCS-BEC crossover tuned by a Feshbach resonance

\[ H = \sum_{p, \sigma} (\varepsilon_p - \mu)c_{p, \sigma}^\dagger c_{p, \sigma} - U \sum_{p, p', q} c_{p+q, -q}^\dagger c_{p', -q}^\dagger c_{p', q}\downarrow c_{p, q}\uparrow \]

\[ + \sum_{q} (\varepsilon_q^B + 2\nu - 2\mu)b_q^\dagger b_q + g \sum_{p, q} \left[ b_q^\dagger c_{-p+q/2, -q}^\dagger c_{p, +q/2}\uparrow + c_{p+q/2, +q/2}\uparrow c_{-p+q/2, q}\downarrow b_q \right] \]

The fermion field and boson field lead to the two superfluid order parameters:

\[ \Delta = U \sum_p \langle BCS | c_{k, \uparrow}^\dagger c_{-k, \downarrow}^\dagger | BCS \rangle \quad \text{BCS order parameter} \]

\[ \phi_m = \langle b_0 \rangle = \langle b_0^\dagger \rangle \quad \text{BEC molecular condensate} \]

However, these are NOT independent, but related to each other due to the resonance between atoms and molecules.

\[ \phi_m = -\frac{1}{2\nu - 2\mu U} \frac{g}{\Delta} \]

\[ \left( 0 = \frac{\partial}{\partial t} \langle b_0 \rangle = \frac{i}{\hbar} \langle [b_0, H] \rangle = (2\nu - 2\mu)\phi_m + \frac{g}{U} \Delta \right) \]

Thus, both order parameters are finite in the superfluid phase.
The mean field CFB Hamiltonian has the form

\[ H = \sum_{p\sigma} (\varepsilon_p - \mu) c_{p\sigma}^\dagger c_{p\sigma} - \Delta \sum_p \left( c_{p\uparrow}^\dagger c_{-p\downarrow}^\dagger + c_{-p\downarrow} c_{p\uparrow} \right) + \sum_{q \neq 0} (\varepsilon_q^B + 2\nu - 2\mu) b_q^\dagger b_q \]

BCS-type with composite order parameter

\[ \Delta = \Delta - g \phi_m \]

Free Bose gas

\[ E_p = \sqrt{(\varepsilon_p - \mu)^2 + \Delta^2} \quad \text{: Single-particle excitations} \]

\[ \Delta = U \sum_p \langle BCS | c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger | BCS \rangle = U \sum_p \frac{\Delta}{2\sqrt{(\varepsilon_p - \mu)^2 + \Delta}} \]

\[ 1 = (U + \frac{g^2}{2\nu - 2\mu}) \sum_p \frac{1}{2\sqrt{(\varepsilon_p - \mu)^2 + \Delta}} \quad \text{: gap equation for } \Delta \]
The equation indicates that the effective pairing interaction associated with the Feshbach resonance is given by

\[ U_{\text{eff}} = U + \frac{g^2}{2\nu - 2\mu} \]

Note that this expression is different from the two-particle result:

\[ U_{\text{eff}}^{2b} = U + \frac{g^2}{2\nu} \]

At T=0, thermally excited molecules are absent, so that the equation for the total number of Fermi atoms is given by

\[ N = 2|\phi_m|^2 + \sum_p [1 - \frac{\varepsilon_p - \mu}{\sqrt{(\varepsilon_p - \mu)^2 + \Delta}}] \]
Broad Feshbach resonance: \( g\sqrt{n} \gg \varepsilon_F \)

crossover region \( 2\nu \gg 2\mu \sim 2\varepsilon_F \)

\[
U_{\text{eff}} = U + \frac{g^2}{2\nu - 2\mu} \sim U + \frac{g^2}{2\nu}
\]

\( \phi_m \sim 0 \)

\[
1 = -\frac{4\pi a_s}{m} \sum_p \left[ \frac{1}{2\sqrt{(\varepsilon_p - \mu)^2 + \Delta}} - \frac{1}{2\varepsilon} \right]
\]

\[
N = 2 |p_m|^2 + \sum_p \left[ 1 - \frac{\varepsilon_p - \mu}{\sqrt{(\varepsilon_p - \mu)^2 + \Delta}} \right]
\]

Single-channel BCS model with \( \Delta \)
Narrow Feshbach resonance: \( g \sqrt{n} < \epsilon_F \)

crossover region \( \rightarrow 2\nu \sim 2\mu \)

In this case, one cannot eliminate the cutoff by using the two-body scattering length, because \( U_{\text{eff}} \) is different from \( U_{\text{eff}}^{2b} \).

The cutoff can be formally eliminated by introducing the ‘generalized scattering length,’ defined by

\[
\frac{4\pi a_s}{m} = -\frac{U + \frac{g^2}{2\nu - 2\mu}}{1 - (U + \frac{g^2}{2\nu - 2\mu}) \sum_p \frac{1}{2\epsilon}} \quad \rightarrow \quad 1 = -\frac{4\pi a_s}{m} \sum_p \left[ \frac{1}{2\sqrt{(\epsilon_p - \mu)^2 + \Delta}} - \frac{1}{2\epsilon} \right]
\]

\[
N = 2 \left| \phi_m \right|^2 + \sum_p \left[ 1 - \frac{\epsilon_p - \mu}{\sqrt{(\epsilon_p - \mu)^2 + \Delta}} \right]
\]
Chemical potential in the crossover region

\[ \mu = \nu \]

The pairing interaction \( U_{\text{eff}} \) is always **attractive** and become strong as one decreases the threshold energy \( 2\nu \).

In the strong-coupling regime, \( 2\mu = 2\nu \) is obtained. This means that the molecular chemical potential \( 2\mu \) is at the lowest boson energy (\( = \text{BEC condition} \)).
Composite order parameter in the crossover region

\[ \frac{\Delta}{\epsilon_F} \]

\[ \frac{\Delta}{\epsilon_F} \]

\[ \frac{\Delta}{\epsilon_F} \]

\[ \frac{\Delta}{\epsilon_F} \]

\[ \phi_m / \epsilon_F \]

\[ \phi_m / \epsilon_F \]

\[ \nu / \epsilon_F \]

\[ \nu / \epsilon_F \]

narrow resonance

\[ Un = 0.02 \epsilon_F, g \sqrt{n} = 0.17 \epsilon_F \]

broad resonance

\[ Un = 0.02 \epsilon_F, g \sqrt{n} = 5 \epsilon_F \]
narrow

broad

single-channel
narrow

broad

single-channel

$N_F$

$n_0 + \phi_m^2$

$\phi_m^2$

$n_0$

$N_F$

$n_0 + \phi_m^2$

$\phi_m^2$

$n_0$

BCS $(k_F a_s)^{-1}$ BEC
As far as we consider quantities independent of the character of molecules (Cooper pairs or Feshbach molecules), the narrow FR and broad FR almost give the same results when scaled by the (generalized) scattering length.
Chapter 8.
BCS-BEC crossover at finite temperatures
1. Superfluid phase transition
Breakdown of the mean-field theory at $T>0$

1. Single-channel BCS model

$$
\Delta = U \sum_p \langle c_{p\uparrow}^\dagger c_{-p\downarrow}^\dagger \rangle = U \sum_p \frac{1}{Z} \text{Tr}[e^{-\beta H} c_{p\uparrow}^\dagger c_{-p\downarrow}^\dagger] \quad \Rightarrow \quad 1 = U \sum_p \frac{1}{2E_p} \tanh \frac{\beta E_p}{2}
$$

$$
N = \sum_{p\sigma} \langle c_{p\sigma}^\dagger c_{p\sigma} \rangle = \sum_p \left[ 1 - \xi_p \tanh \frac{\beta E_p}{2} \right] \left( E = \sqrt{\xi^2 + \Delta^2} \right)
$$

$\triangleright\ T_c \Rightarrow \Delta = 0$

$$
1 = -\frac{4\pi a_s}{m} \sum \left[ \frac{1}{2(\varepsilon - \mu)} \tanh \frac{\beta(\varepsilon - \mu)}{2} - \frac{1}{2\varepsilon} \right]
$$

$$
N = 2 \sum_p f(\varepsilon_p - \mu) \quad \text{Free Fermi gas!}
$$

$$
\mu \sim \varepsilon_F \quad (T << T_F)
$$
Breakdown of the mean-field theory at T>0
1. Single-channel BCS model

In the weak-coupling BCS limit, T_c<< T_F, and the equation of states gives \( \mu = \varepsilon_F \), as expected. In this case, the gap equation gives

\[
T_c = \frac{8\gamma}{\pi e^2} \varepsilon_F e^{\frac{\pi}{2k_F a_s}} = 0.61\varepsilon_F e^{\frac{\pi}{2k_F a_s}}
\]

Note: \( \Delta(T = 0) = \frac{8}{e^2} \varepsilon_F e^{\frac{\pi}{2k_F a_s}} \rightarrow 2\Delta / T_c = \pi / \gamma = 3.54 \)  
(BCS universal constant)

In the strong-coupling limit, the gap equation gives

\[
\mu = -\frac{1}{2ma_s^2}
\]

This equals the binding energy of a Cooper-pair obtained at T=0. This means that molecules does not dissociate even at Tc. Thus, one expects that Tc is given by an ideal molecular Bose gas (\( N_B = N/2 \), M=2m) as

\[
T_c = \frac{2\pi}{(\zeta(3/2))^{2/3}} \frac{(n/2)^{2/3}}{(2m)} = 0.218\varepsilon_F
\]

\[
N = 2 \sum_p f(\varepsilon_p - \mu)
\]
2. Coupled Fermion-Boson model

\[
1 = \left( U + \frac{g^2}{2\nu - 2\mu} \right) \sum_p \frac{1}{2\xi_p} \tanh \frac{\beta \xi_p}{2}
\]
\[
N = 2 \sum_q n_B(\epsilon_b^B + 2\nu - 2\mu) + 2 \sum_p f(\epsilon_p - \mu) = 2N_{B0} + N_{F0}
\]

**Weak-coupling regime** \((2\nu >> 2\mu)\):

\[N_{B0} = 0 \Rightarrow \mu \sim \epsilon_F \Rightarrow BCS\]

**Strong-coupling regime** \((2\nu \sim 2\mu < 0)\):

\[T_c = \frac{2\pi}{\zeta(3/2)^{2/3}} \frac{(n/2)^{2/3}}{(2m)} = 0.218\epsilon_F\]

**Crossover region**:

\[2\nu > 2\mu\]

A finite energy gap exists in the molecular excitations even at \(T_c\).
Strong-coupling theory (Nozieres and Schmitt-Rink: NSR)

Importance of thermal fluctuations in the Cooper-channel to describe the BCS-BEC crossover at T>0.

NSR = Thouless criterion + Gaussian fluctuation

\[ H = \sum_{p\sigma} \left( \varepsilon_p - \mu \right) c_{p\sigma}^\dagger c_{p\sigma} - U \sum_{p,p',q} c_{p+q\uparrow}^\dagger c_{p'\downarrow}^\dagger c_{p'\downarrow} c_{p\uparrow} \]
**Thouless criterion**

\[ \Gamma = U + \ldots \]

Fermion: \( G_{p\sigma}(i\omega_n) = \frac{1}{i\omega_n - \xi_p} \)

\[ 1 = U \sum \frac{1}{2(\varepsilon - \mu)} \tanh \frac{\beta(\varepsilon - \mu)}{2} \]

\( \text{pole of } \Gamma \text{ at } q=\omega=0 \rightarrow T_c \)
μ: fluctuation contribution to the thermodynamic potential

\[ \Omega = \Omega_{\text{MF}} + U \]

Fluctuations in the Cooper-channel

\[ N = N_{F0} - T \frac{\partial}{\partial \mu} \sum_{n,\nu_n} e^{i\delta\nu_n} \log[1 - U\Pi(q, iv_n)] \]

\[ \Pi(q, iv_n) = \frac{1}{\beta} \sum_{p,\omega_n} G_{p+q/2, \uparrow}(i\omega_n + iv_n)G_{-p+q/2, \downarrow}(-i\omega_n) \]

(pair fluctuation propagator in the Cooper-channel)

\[ N_{F0} = 2 \sum f(\varepsilon_p - \mu) \]
\[ 1 = -\frac{4\pi a_s}{m} \sum_p \left[ \frac{1}{2(\varepsilon_p - \mu)} \tanh \frac{\beta(\varepsilon_p - \mu)}{2} - \frac{1}{2\varepsilon_p} \right] \]

\[ N = 2\sum_p f(\varepsilon_p - \mu) - T \left\langle \frac{\partial}{\partial \mu_{q,\pi_n}} \sum e^{i\delta_{\pi_n}} \log \left[ 1 + \frac{4\pi a_s}{m} \left( \Pi(q,iv_n) - \sum_p \frac{1}{2\varepsilon_p} \right) \right] \right\rangle \]

Tc and \( \mu \) are determined self-consistently for a given U.
NSR Theory

(BCS limit: \((k_F a_s)^{-1} \ll -1\))

\[
1 = -\frac{4\pi a_s}{m} \sum_p \left[ \frac{1}{2(\varepsilon_p - \mu)} \tanh \frac{\beta(\varepsilon_p - \mu)}{2} - \frac{1}{2\varepsilon_p} \right]
\]

\[
N = 2 \sum_p f(\varepsilon_p - \mu) - T \frac{\partial}{\partial \mu} \sum_{q,\nu_n} e^{i\delta_{\nu_n}} \log \left[ 1 + \frac{4\pi a_s}{m} \left[ \Pi(q, iv_n) - \sum_p \frac{1}{2\varepsilon_p} \right] \right]
\]

\[
\mu = \varepsilon_F \quad (T_c \ll T_F)
\]
Expanding around \( \omega=q=0 \), we obtain

\[
\frac{N}{2} = \frac{1}{\beta} \sum_{q,\nu_n} \frac{1}{q^2} e^{i\nu_n} = \sum_b n\left(\frac{q^2}{2m} - \mu_B\right)
\]

condition for BEC of N/2 Bose gas.

\[
\mu_B = \sqrt{\frac{\mu}{2ma_s^2}} \left[1 - \sqrt{2ma_s^2} |\mu| \right] \Rightarrow 0
\]
NSR Theory: numerical result

\[ T_c/\epsilon_F = 0.218 T_F \]

BCS \hspace{1cm} BEC

\[ \frac{\mu}{\epsilon_F} = -\frac{1}{2ma_s^2} \]

Extension to the coupled Fermion-Boson model

Two hyperfine states: \( \uparrow \downarrow \)

\[
H = \sum \varepsilon_p c_p^+ c_p - \sum E_q b_q^+ b_q + g \sum \left( b_q^+ c_{-p+q/2} + h.c. \right) + U \sum c_p^+ c_{-p} c_{-p}^+ + V_{trap}
\]

\[
E_q = \frac{q^2}{2M} + 2\nu
\]

Feshbach resonance

\[ N_{Fermi} + 2N_{Boson} = N \quad \rightarrow \quad H - \mu N \]
Extension to the coupled Fermion-Boson model

Thouless criterion

\[ \Gamma = U + \text{Fermion:G} \]

(resonance) Boson: \( D_0 \)

Feshbach resonance

pole of \( \Gamma \) at \( q=\omega=0 \)

\[ 1 = \left( U + \frac{g^2}{2\nu - 2\mu} \right) \frac{\tanh \frac{\beta}{2} (\varepsilon_p - \mu)}{2(\varepsilon_p - \mu)} \]

Interaction associated with a Feshbach resonance
Extension to the coupled Fermion-Boson model

\( \Omega = \Omega_{MF} + U \)  

Nozieres and Schmitt-Rink

Feshbach resonance

\[ N = -\frac{\partial \Omega}{\partial \mu} = N_F^0 + N_B^0 - T \sum_q \log \left[ 1 - U_{\text{eff}}(q)\Pi(q) \right] \]

\[ U_{\text{eff}}(0) = U + \frac{g^2}{2\nu - 2\mu} \]

\[ U_{\text{eff}}(q) = U - g^2 D_0(q) \]
Extension to the coupled Fermion-Boson model

\[
N = N_F^0 + N_B^0 - T \sum_q \log \left[ 1 - U_{\text{eff}}(q) \Pi(q) \right]
\]

\[
= N_F^0 + N_B - T \sum_q \log \left[ 1 - U \Pi(q) \right]
\]

\[
N_B = -T \frac{\partial}{\partial \mu_q} \sum_{q,n} e^{i\delta_{q,n}} \log \left[ iv_n - (\varepsilon^B_q + 2\nu - 2\mu) + g^2 \frac{\Pi(q,iv_n)}{1 - U \Pi(q,iv_n)} \right]
\]

\[
D(q,iv_n) = \frac{1}{iv_n - (\varepsilon^B_q + 2\nu - 2\mu) + g^2 \frac{\Pi(q,iv_n)}{1 - U \Pi(q,iv_n)}}
\]

**Gap eq. at Tc:** \( 2\nu - 2\mu = g^2 \frac{\Pi(0,0)}{1 - U \Pi(0,0)} \)

Molecular excitations are gapless at \( q=0 \)!
Extension to the coupled Fermion-Boson model: numerical result

\[ 0.218T_F \]

\[ \frac{T_c}{T_F} \]

BCS BEC

\[ 1/k_F a \]

Chapter 9.
BCS-BEC crossover at finite temperatures
2. Superfluid phase below $T_c$
Extended NSR in the superfluid phase below $T_c$

\[ H = \sum_{p\sigma} (\varepsilon_p - \mu) c_{p\sigma}^+ c_{p\sigma} - \sum_{p, p', q} c_{p+q\uparrow}^+ c_{p'q\downarrow}^+ c_{p\downarrow} c_{p\uparrow} \]

\[ = \sum_{p\sigma} (\varepsilon_p - \mu) c_{p\sigma}^+ c_{p\sigma} - \Delta \sum_{p} (c_{p\uparrow}^+ c_{-p\downarrow}^+ + c_{-p\downarrow} c_{p\uparrow}) \quad \text{MF (BCS) part} \]

\[ -\frac{U}{4} \sum_{q} [{\rho_1(q)\rho_1(-q) + \rho_2(q)\rho_2(-q)}] \quad \text{Fluctuation part} \]

\[ \rho_j(q) = \sum_p \Psi_{p+q\uparrow}^+ \tau_j \Psi_p \quad \text{generalized density operator} \]

\[ \begin{cases} \rho_1 & : \text{amplitude fluctuations} \\ \rho_2 & : \text{phase fluctuations} \end{cases} \]

\[ \Psi_p = \begin{pmatrix} c_{p\uparrow}^+ \\ c_{-p\downarrow}^+ \\ c_{-p\downarrow} \end{pmatrix} : \text{Nambu field} \quad \tau_j : \text{Pauli matrix} \]
Extended NSR in the superfluid phase below $T_c$

gap equation (below $T_c$)

$$1 = U \sum_p \frac{1}{2E_p} \tanh \frac{E_p}{2T}$$

chemical potential

$$\partial \Omega = \sum_{i,j=1,2} \Pi_{ij} + \cdots$$

$$\Pi_{ij}(q, i\nu_n) = T \sum_{p,\omega_n} \text{tr} \left[ \tau_i G(p+q/2, i\omega_n + i\nu_n) \tau_j G(p-q/2, i\omega_n) \right]$$

$$G(p, i\omega_n) = \frac{1}{i\omega_n - \xi_p \tau_3 - \Delta \tau_1}$$

$$N = N_F - \frac{T}{2} \sum_{q,\nu_n} \frac{\partial}{\partial \mu} \text{tr} \log \left[ 1 + \frac{1}{2} U \hat{\Pi}(q, i\nu_n) \right]$$

$$\hat{\Pi}(q) = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}$$
BCS-BEC crossover in the superfluid phase below $T_c$
Collective modes associated with phase fluctuations appear in the phase correlation function:

\[
\tilde{\Pi}_{22}^{RPA}(q, \omega + i\delta) = \frac{\Pi_{22}^{RPA}(q, \omega + i\delta)}{1 + (U/2)\Pi_{22}^{RPA}(q, \omega + i\delta)}
\]

\[
\Pi_{22}^{RPA}(q, \omega + i\delta) = \Pi_{22} - \Pi_{12} \frac{U/2}{1 + (U/2)\Pi_{11}} \Pi_{21}
\]

(NOTE: RPA is consistent with the extended NSR theory.)
Goldstone mode at finite temperatures

\[
S(q, \omega + i\delta) = -\frac{1}{\omega} \text{Im} \left[ \tilde{\Pi}^{RPA}_{22}(q, \omega + i\delta) \right]
\]

\[\left(\frac{\omega}{\varepsilon_F}\right)^{-1} = -2\quad \text{BCS}\]

\[\left(\frac{\omega}{\varepsilon_F}\right)^{-1} = 0\quad \text{Crossover}\]

\[\left(\frac{\omega}{\varepsilon_F}\right)^{-1} = 2\quad \text{BEC}\]
Goldstone mode at finite temperatures

\[ v_\phi(T = 0) \]

\[ q / k_F = 0.005 \]
Bogoliubov phonon in Boson BEC (Na)

Andrews et al. (1997)

A similar experiment would be also useful for Fermi gases.
Superfluid density in the crossover region

The superfluid density is the most fundamental quantity in the two-fluid hydrodynamics. It is always equal to the total atom density at $T=0$, and equal to 0 at $T_c$.

\[
n = \rho_s + \rho_n
\]

\[
\rho_s = N + \frac{1}{m} \frac{\partial^2}{\partial \nu_s^2} (\Omega_{MF} + \Omega_{FL})_{\nu_s \rightarrow 0}
\]

\[
\tilde{G}(\mathbf{p}, i\omega_n) = \frac{1}{i\omega_n - \mathbf{v}_s \cdot \mathbf{p} - \xi_p \tau_3 - \Delta \tau_1}
\]

Flucuation contribution is crucial so as to satisfy the general requirement of the superfluid density at $T=0$ and $T=T_c$. 

\[
\Delta \rightarrow \Delta e^{i\mathbf{q} \cdot \mathbf{r}}
\]
Superfluid density in the crossover region

\[ \rho_s / n \]

\[ \frac{N_S}{N} \]

\[ T / T_F \]

\[ (k_F a_s)^{-1} \]

\[ n_0 / n \]

Superfluid density

Condensate fraction

carrier density contributing to the supercurrent

the number of Bose-condensed Cooper-pairs
Comparison with experiment (condensate fraction)

The graph shows the comparison of theoretical predictions with experimental data for the condensate fraction of a system described by the equation $N_0/N$. The plot includes data for $^{40}$K, with curves labeled BCS (Bardeen-Cooper-Schrieffer) and BEC (Bose-Einstein Condensate). The axes represent $T/T_F$ vs. $1/k_Fa$, with color coding indicating the value of $N_0/N$. The graph illustrates the transition from BCS to BEC behavior as the temperature decreases.
Origin of the first order phase transition

Condensate fraction

\[ \frac{n_0(T)}{n_0(0)} \]

(a)

\[ \frac{n_0}{n} \]

(b)

The result in the BEC regime is well described by the Popov theory for a weakly interaction Bose gas with \( a_B = 2a_s \).
Chapter 10.
Effective interaction between Cooper pairs in the BEC regime
There exists a weakly repulsive interaction between ‘Cooper-pair bosons’. This interaction is important to stabilize the superfluid phase.

Boson BEC:  \[ E_q = \sqrt{\varepsilon_q (\varepsilon_q + 2Un_0)} \]

When \( U < 0 \) (attractive), the excitation spectrum unphysically has an imaginary part in the long-wavelength limit!

\[ U_B = \frac{4\pi a_B}{M} \]

What is the origin of this interaction??
Interaction between Cooper-pairs and effects of single-particle excitations in the BEC regime

BEC regime can be well described by tightly bound molecules

For a uniform case, $\mu$ is given by

$$\mu = -1/(2ma_s^2) \quad \text{(BEC region)}$$

Thus, the binding energy of a molecule becomes very large in the BEC limit:

$$2E_g = 2 | \mu | \rightarrow \infty \quad (a_s \rightarrow +0)$$

At a glance, $E_g$ seems irrelevant as far as we consider the low energy physics. However, we show that even in the BEC regime, the energy gap $E_g$ still plays an important role.
Effective interaction between Cooper-pair molecules is mediated by *dissociated* Fermi atoms.

\[ V_{\text{eff}} = \frac{4\pi a_B}{M_M} = 2E_g \]

- \( a_B = 2a_s \)  
  - NSR
- \( a_B = 0.6a_s \)  
  - 2-molecule (4-atom) analysis (Petrov et al., 2004)
- \( a_B = 0.75a_s \)  
  - 2-particle t-matrix analysis (Strinati et al., 2000)
\[ V_{\text{eff}} = \frac{4\pi(2a_s)}{M_B} \]

- \( q \) and \(-q\)

- cutoff energy of molecular interaction

- Binding energy of a Cooper-pair molecule

\[ \omega_c = \left( \frac{8}{5} \right) \mu \]

\[ 2E_g = 2 \mu \]
We have to include the effect of the **physical cutoff** $\omega_c$, originating from the binding energy $2E_g$ of a Cooper-pair.

$$H_\phi = \sum \left( \frac{q^2}{2M_B} - \mu_B \right) \phi_q^\dagger \phi_q + \frac{1}{2} \frac{4\pi (2a_s)}{M_B} \sum_{\omega_c} \phi_p^\dagger \phi_{p+q}^\dagger \phi_{p-q} \phi_p \phi_q$$

$$\omega_c = \left( \frac{8}{5} \right) | \mu | \sim 2E_g \ , \ (E_g = | \mu |)$$

$$\mu = -\frac{1}{2ma_s^2} \quad \text{(BEC limit)}$$

$\phi_q$ : Bose field (Cooper-pairs)

**(NOTE: This effective Hamiltonian is valid for $(k_Fa_s)^{-1} > 1 : \text{BEC}$**

$a_B=2a_s$ should be regarded as a **bare** scattering length at the cutoff $\omega_c$. 
Two-molecule case

The low-energy two-particle molecular scattering length is given by

\[ U_{\text{eff}}(0) = \frac{4\pi a^2_B}{M_B} = \frac{4\pi(2a_s)}{1 + \frac{M_B}{4\pi(2a_s)} \sum_{q}^{\omega_c} \frac{1}{q^2} 2\cdot q^2 2M_B} \]

\[ = \frac{4\pi(0.61a_s)}{M_B} \]

\[ a^2_B = 0.61a_F \]

Petrov et al.

Strinati et al.

The binding energy \(2E_g\) is the key to obtain the low-energy molecular scattering length.
Molecular interaction in the strong-coupling BEC regime [Many particles, T>0]

Many-body effect and the effect of $\omega_c$, can be taken into account by using the renormalization group.

Two-particle result:

$$a_B^{2b} = 0.6a_s$$

$T_c$ is within the 1-loop level.
Chapter 11.
Effective of a trap potential
Coupled Fermion-Boson model in a trap

Superconductivity: T.D. Lee, Ranninger, Fermi atom gas: Timmermans, Holland, Ohashi, and their collaborators

two atomic hyperfine states = \[ \downarrow \uparrow \]

\[
H = \sum_{\sigma} \int dr \Psi_\sigma^\dagger (r) \left[ \frac{p^2}{2m} + V_{\text{Fermion}}^{\text{trap}} (r) \right] \Psi_\sigma (r) - U \int dr \Psi_\uparrow^\dagger \Psi_\downarrow^\dagger \Psi_\downarrow \Psi_\uparrow \\
+ \Phi^\dagger (r) \left[ \frac{p^2}{2M} + 2\nu + V_{\text{Boson}}^{\text{trap}} (r) \right] \Phi (r) \\
+ g \sum_{\sigma} \int dr [ \Phi^\dagger (r) \Psi_\downarrow (r) \Psi_\uparrow (r) + h.c. ]
\]

Feshbach molecule

Feshbach resonance

\[ N = N_{\text{atom}} + 2N_{\text{molecules}} \]

Feshbach molecule

Fermi atom

\[ H \rightarrow H - \mu N \]
BCS-BEC crossover theory at T=0 (absence of thermal fluctuations)

\[ H_{HFB} = \sum_{\sigma} \int dr \Psi_\sigma^\dagger(r) \left[ \frac{p^2}{2m} - \mu - \frac{U}{2} n_F(r) + V_{\text{Fermion}}^{\text{trap}}(r) \right] \Psi_\sigma(r) \]

\[ - \int dr [\Delta(r) \Psi_\downarrow(r) \Psi_\uparrow(r) + h.c.] \]

\[ + \int dr \delta \Phi^\dagger(r) \left[ \frac{p^2}{2M} + \nu - 2\mu + V_{\text{Boson}}^{\text{trap}}(r) \right] \delta \Phi(r) \]

Bogoliubov de-Gennes equations

\[ N(\mu) = N_{\text{atoms}} + 2N_{\text{molecules}} \]

\( (\Delta, n_F, \mu) \) are determined self-consistently.
Atom density profile in the BCS-BEC crossover region at $T=0$

$$g\sqrt{n} = 10\varepsilon_F$$

$F$-gas

$\varepsilon_F = \frac{1}{2}(k_Fa_s)^{-1} = +0.67$

$\varepsilon_F = \frac{1}{2}(k_Fa_s)^{-1} = -1.2$

$R_F$: Thomas Fermi radius

Fermi atoms form Cooper-pairs; however, they do not affect the density profile in the BCS regime.

Density profile shrinks in the BEC regime, due to bosonic character of tightly-bound Cooper pairs.
superfluid density of states (trapped gas)

Sharp peaks are due to discrete levels in a trap.

**uniform superfluid:**  
\[ E = \sqrt{\left(\varepsilon - \mu\right)^2 + \Delta^2} \]

\[ \Rightarrow E_g = \tilde{\Delta} \quad (\mu > 0) \]

\[ \Rightarrow E_g = \sqrt{\mu^2 + \Delta^2} \rightarrow |\mu| \quad (\mu < 0) \]

(|\mu| >> \Delta)
Local density of states

The lowest Andreev bound state determines the excitation gap $E_g$. 
rf-tunneling current spectroscopy

BCS BEC unitarity limit

\[ I(\omega) \]

\[ \omega [\omega_0] \]

\[ \mu > 0 \]

\[ \mu < 0 \]

\[ \text{BEC} \]

\[ \text{unitarity limit} \]

\[ \text{BCS} \]

\[ 6^\text{Li}: \text{Chin et al., Science 305 1128 (2004)}. \]

\[ \psi_a \]
Peak energy gives $\Delta(r = 0)$. The rf-tunneling measurement can also detect $E_g$ from the threshold energy (but very high resolution is required).

Experiment on $^6$Li: Chin et al., Science 305 1128 (2004).
Chapter 12.
BCS-BEC crossover in a lattice system
Standing wave of laser light: \[ E(x) = E_0 \sin (kx)e^{i\alpha} \]

Stark Effect: energy shift of atoms by s-p dipole transition

\[ \Delta E = -\frac{\alpha}{2} |E(x)|^2 \propto \sin^2 (kx) \Leftrightarrow V(x) \]

Fermi gas in an optical lattice

Kohl et al., PRL 94 (2005) 080403

Observation of the Brillouin zone
A Fermi gas in an optical lattice is described by the Hubbard model (when lattice potential is strong enough).

\[ H = -t \sum_{\langle ij \rangle, \sigma} \left[ c_{i\sigma}^\dagger c_{j\sigma} + h.c. \right] - U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i, \sigma} n_{i\sigma} \]
BCS-BEC crossover at $T_c$ in an optical lattice

The binding energy in the BEC regime equals $U$. 

Lattice

Uniform gas

The binding energy in the BEC regime equals $U$. 

$n=1.0$

$BCS$ limit at $n=1.0$

$BEC$ limit at $n=1.0$
BCS-BEC crossover at $T_c$ in an optical lattice

Hopping of a Cooper pair is accompanied by dissociation.

Binding energy $\sim U$

(BEC)

Effective hopping of a molecule $\sim \frac{t^2}{U}$

Effective mass of a Cooper pair $M \sim U$

$$T_c = \frac{\frac{2\pi}{\zeta(3/2)} (\frac{n}{2})^{2/3}}{M} \propto \frac{1}{U}$$